

ON THE CONVERGENCE OF THE SASAKI J -FLOW

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ABSTRACT. This paper investigates the C^∞ -convergence of the Sasaki J -flow. The result is applied to prove a lower bound for the K -energy map in the Sasakian context.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The Sasaki J -flow has been introduced in [23] as a natural counterpart of the Kähler J -flow in the Sasakian setting. In [8, pages 10-11] Donaldson introduced the Kähler J -flow on a n -dimensional Kähler manifold M pointing out the importance of its critical points from the point of view of moment maps. Given two Kähler forms ω and χ on M , such critical points satisfy the equation $n\chi \wedge \omega^{n-1} = c\omega^n$, where c is a constant depending on $[\omega]$ and $[\chi]$. Donaldson observed that a necessary condition for their existence is that $[c\omega - \chi]$ be a Kähler class, and he asked if it was also sufficient. In [5] X.X. Chen proved that for complex surfaces it is. Although, it is not in higher dimension, indeed in the recent paper [15], M. Lejmi and G. Székelyhidi found a counterexample on the blow up of \mathbb{P}^3 at one point. The existence of critical points has been studied by B. Weinkove, V. Tosatti and J. Song in [19, 22] in terms of the positivity of some $(n-1, n-1)$ form and by X.X. Chen in [6] in terms of the sign of the holomorphic sectional curvature of χ . In particular, in [6] Chen proved the long time existence of the flow and, when the bisectional curvature of χ is nonpositive, its convergence to a critical metric.

This work was inspired by [24], where B. Weinkove deals with the natural question on what the behaviour of the flow is on Kähler surfaces, where the existence of a critical metric is always guaranteed (once the necessary condition above is satisfied). He proved the convergence of the J -flow on Kähler surfaces under the only assumption $c\omega - \chi$ to be positive. In the later work [25], Weinkove generalize his result to higher dimension, proving the convergence of the Kähler J -flow under the assumption of the positivity of the form $c\omega - (n-1)\chi$.

In the Sasakian case the situation is quite similar and a necessary condition for the existence of a critical point is that there exists a basic map h such that $\frac{c}{2}(d\eta + dd^c h) - \chi$ is a transverse Kähler form (see [23] or next section below for definitions and details). In [23, Prop. 3.3], it is proven that that condition is also sufficient on 5-dimensional Sasakian manifolds.

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The first result of this paper is the following theorem, which translates Weinkove's result in [25] to the Sasakian context (we remaind to Section 2 for definitions and details):

Theorem 1.1. *Let (M, ξ, Φ, η, g) be a compact $2n + 1$ -dimensional Sasaki manifold. Assume that $\frac{c}{2}d\eta - (n - 1)\chi$ is a transverse Kähler form. Then, the Sasaki J -flow converges C^∞ to a smooth critical metric.*

The proof is based on the long time existence of the flow and the uniform lower bound on the second derivatives of a solution to the flow established in [23], and on the estimates developed in sections 3 and 4 of the present paper, which are obtained applying the maximum principle and a Moser iteration argument (see the proof of Prop 4.6).

It is worth pointing out that as immediate corollary we get the C^∞ convergence of the flow to a critical metric on compact 5-dimensional Sasaki manifolds under the assumption $\frac{c}{2}d\eta - \chi > 0$.

As application, we highlight the relation between the Sasaki J -flow and the Mabuchi K -energy, introduced in the Sasakian context by [9] (see also [14]), proving a lower bound for the K -energy map under the existence of a critical metric (see Theorem 5.2 at the end of the paper).

The paper contains three more sections. In the first one we summarize some basic facts about Sasakian geometry, recall the definition of Sasaki J -flow and set the notations. In the second one we develop second order estimates on a solution f to the J -flow which depends on f itself. In the third section we study the C^0 -estimates and prove Theorem 1.1. Finally in the fourth and last section we recall the definition of Mabuchi K -energy in the Sasakian context and prove our second result Theorem 5.2.

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2. SASAKIAN MANIFOLDS AND THE SASAKI J -FLOW

Here we briefly recall what we need about Sasakian manifolds, the reader is referred to [2, 20] for a more detailed exposition.

A Riemannian manifold (M, g) is Sasakian if and only if the Riemannian cone $(M \times \mathbb{R}^+, \bar{g} = r^2g + dr^2)$ is Kähler. The integrable complex structure J and the Kähler form $\bar{\omega}$ on $(M \times \mathbb{R}^+, \bar{g})$ induce in a natural way on (M, g) :

- (1) a killing vector field ξ and its dual 1-form η , $\eta(\xi) = 1$, $\iota_\xi d\eta = 0$, which is a contact form, i.e. $\eta \wedge (d\eta)^n \neq 0$. The tangent bundle TM splits into $TM = D \oplus L_\xi$, where $D = \ker \eta$ and L_ξ is the line tangent to ξ .
- (2) an endomorphism Φ defined by $\Phi|_D = J|_D$, $\Phi_{L_\xi} = 0$, which satisfies $\Phi^2 = -\text{Id} + \eta \otimes \xi$.

The triple (η, ξ, Φ) realises a contact structure on M , while $(D, \Phi|_D)$ realises a CR structure.

According to $TM = D \oplus L_\xi$, the metric g splits into:

$$g(X, Y) = g^T(X, Y) + \eta(X)\eta(Y), \quad X, Y \in TM,$$

where $g^T(X, Y) = \frac{1}{2}d\eta(X, \Phi Y)$, which is zero along the direction of ξ and it is Kähler with respect to D , is called the *transverse Kähler metric* of M .

A p -form α on M is basic if it satisfies:

$$\iota_\xi \alpha = 0, \quad \iota_\xi d\alpha = 0.$$

In particular, a function is basic if and only if its derivatives in the direction of ξ vanishes. We denote the space of smooth basic functions on M by $C_B^\infty(M, \mathbb{R})$.

Given a Sasakian manifold (M, g, ξ, Φ, η) , consider

$$\mathcal{H} = \{f \in C_B^\infty(M, \mathbb{R}) \mid \eta_f = \eta + d^c f \text{ is a contact form}\},$$

where $(d^c f)(X) = -\frac{1}{2}df(\Phi(X))$ for any vector field X on M . Observe that any $f \in \mathcal{H}$ induces a Sasakian structure (ξ, Φ_f, η_f) on M with the same Reeb vector field ξ . The geometry of \mathcal{H} has been studied by P. Guan and X. Zhang in [10, 11] from the point of view of geodesics, by W. He in [13] from the point of view of curvature and by S. Calamai, D. Petrecca and K. Zheng in [3] in relation to the Ebin metric.

In order to define the J -flow, we need to fix a transverse Kähler form χ on M , i.e. χ is a basic $(1, 1)$ -form which is positive and closed. Let $f(t)$ be a smooth path on \mathcal{H} . Define the functional $J_\chi: \mathcal{H} \rightarrow \mathbb{R}$ by:

$$(\partial_t J_\chi)(f) = \frac{1}{2^{n-1}(n-1)!} \int_M \dot{f} \chi \wedge \eta \wedge (d\eta_f)^{n-1} = \frac{1}{2^n n!} \int_M \dot{f} \sigma_f \eta \wedge (d\eta_f)^n, \quad J_\chi(0) = 0,$$

where σ_f is the trace of χ with respect to $d\eta_f$. Alternatively, the J_χ functional can be defined by (cfr. [23, Def. 2.2]):

$$J_\chi(h) = \frac{1}{2^{n-1}(n-1)!} A_\chi(0, h)$$

where

$$A_\chi(f) := \int_0^1 \int_M \dot{f} \chi \wedge \eta \wedge (d\eta_f)^{n-1} dt.$$

Further, let $\mathcal{H}_0 = \{h \in \mathcal{H} \mid I(h) = 0\}$, where $I: \mathcal{H} \rightarrow \mathbb{R}$ is defined by:

$$(\partial_t I)(f) = \int_M \dot{f} \eta \wedge (d\eta_f)^n, \quad I(0) = 0.$$

Notice that I can be explicitly written by (see [10, Eq. 14]):

$$(1) \quad I(f) = \sum_{p=0}^n \frac{n!}{(p+1)!(n-p)!} \int_M f \eta \wedge (d\eta_0)^{n-p} \wedge (i\partial_B \bar{\partial}_B f)^p.$$

Observe that $h \in \mathcal{H}_0$ is a critical point of J_χ restricted to \mathcal{H}_0 if and only if

$$\int_M k \eta \wedge \chi \wedge (d\eta_h)^{n-1} = 0,$$

for every $k \in T_h \mathcal{H}_0$, i.e. if and only if $2n \eta \wedge \chi \wedge (d\eta_h)^{n-1} = c \eta \wedge (d\eta_h)^n$, where

$$c = \frac{2n \int_M \chi \wedge \eta \wedge (d\eta)^{n-1}}{\int_M \eta \wedge (d\eta)^n}.$$

In particular, $h \in \mathcal{H}_0$ is a critical point of J_χ iff $\sigma_h = c$. The Sasaki J -flow is the gradient flow of $J_\chi: \mathcal{H}_0 \rightarrow \mathbb{R}$ and its evolution equation is given by:

$$(2) \quad \dot{f} = c - \sigma_f, \quad f(0) = 0.$$

In the joint work with L. Vezzoni [23], we study the long time existence of the J -flow (2) and prove its convergence to a critical metric under an additional hypothesis on the sign of the transverse holomorphic sectional curvature of χ . In the recent paper [1], the Sasaki J -flow is included as particular case in a more general result that prove the short time existence of second order geometric flows on foliated manifolds.

We conclude this section recalling the definition of *special foliated coordinates* (see [23, Subsec. 2.1]) and setting some notations.

Let $(M, \xi, \Phi_f, \eta_f, g_f)$ be a Sasakian manifold as above and let χ be a second transverse Kähler form on M . Around each point $(x_0, t_0) \in M \times [0, +\infty)$ we can find *special foliated coordinates* $\{z^1, \dots, z^n, z\}$ for χ , taking values in $\mathbb{C}^n \times \mathbb{R}$, such that

$$(3) \quad \xi = \partial_z, \quad \Phi(dz^j) = i dz^j, \quad \Phi(d\bar{z}^j) = -i d\bar{z}^j,$$

and if we denote $\chi = \chi_{i\bar{j}} dz^i \wedge d\bar{z}^j$, then

$$\chi_{i\bar{j}} = \delta_{ij}, \quad \partial_{z^r} \chi_{i\bar{j}} = 0, \quad \text{at } (x_0, t_0),$$

and

$$(g_f)_{j\bar{k}} = \lambda_j \delta_{jk}, \quad \text{at } (x_0, t_0).$$

Here we denote:

$$g_f = (g_f)_{i\bar{j}} dz^i d\bar{z}^j + \eta_f^2, \quad d\eta_f = 2i(g_f)_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

and in particular the transverse Kähler metric g_f^T reads locally $g_f^T = (g_f)_{i\bar{j}} dz^i d\bar{z}^j$.

Observe that $(g_f)_{i\bar{j}}$ are basic functions and by (3), in these coordinates a function is basic iff it does not depend on z . Let us also use upper indexes to denote the entries of a matrix' inverse. Further, for any $h \in C_B^\infty(M)$ we denote $f_{,j} = \partial_j f = \frac{\partial}{\partial z_j} f$, $f_{,\bar{j}} = \partial_{\bar{j}} f = \frac{\partial}{\partial \bar{z}_j} f$, $f_{,j\bar{k}} = \partial_j \partial_{\bar{k}} f = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} f$. In the sequel we will also denote by γ_f the trace of g_f with respect to χ , i.e. locally $\gamma_f = \chi^{\bar{j}k} (g_f)_{j\bar{k}}$, and by $R(\chi)_{j\bar{k}l\bar{m}}$ the curvature tensor associated to χ , which in special coordinates for χ at (x_0, t_0) reads $R(\chi)_{j\bar{k}l\bar{m}} = -\chi_{j\bar{k},l\bar{m}}$. Consequently we will have:

$$R(\chi)^{a\bar{b}}_{l\bar{m}} = -\chi^{\bar{a}j} \chi^{\bar{k}b} \chi_{j\bar{k},l\bar{m}}, \quad \text{Ric}(\chi)_{l\bar{m}} = -\chi^{\bar{k}j} \chi_{j\bar{k},l\bar{m}}.$$

Further, we denote by $(\cdot, \cdot)_\chi$ the product on basic forms $\alpha, \beta \in \Omega_B^{(p,q)}(M, \mathbb{C})$:

$$(\alpha, \beta)_\chi = \frac{1}{2^n n!} \int_M \langle \alpha, \beta \rangle_\chi \eta \wedge \chi^n.$$

where

$$\langle \alpha, \beta \rangle_\chi = \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} \cdot \bar{\beta}_{r_1 \dots r_p \bar{s}_1 \dots \bar{s}_q} \chi^{\bar{r}_1 i_1} \dots \chi^{\bar{r}_p i_p} \cdot \chi^{\bar{j}_1 s_1} \dots \chi^{\bar{j}_q s_q}.$$

Finally, let $\tilde{\Delta}_f$ be the operator depending on a smooth curve f in \mathcal{H} and acting on basic smooth functions, defined by:

$$\tilde{\Delta}_f(h) = g_f^{\bar{k}p} g_f^{\bar{q}j} \chi_{j\bar{k}} h_{,a\bar{b}}.$$

Remark 2.1. Observe that the operator $\tilde{\Delta}_f$ satisfies the following property. If $h(x, t)$ is a smooth path in \mathcal{H} and (x_0, t_0) is a global maximum for h in $M \times [0, t]$, then by the maximum principle at (x_0, t_0) one has:

$$(\partial_t - \tilde{\Delta}_f)h \geq 0.$$

More precisely, at (x_0, t_0) one has $\partial_t h \geq 0$ and $dd^c h \leq 0$. Then, by the definition of $\tilde{\Delta}_f$, one get $\tilde{\Delta}_f(h) \leq 0$ and thus $\partial_t - \tilde{\Delta}_f \geq 0$.

Remark 2.2. Observe that in special foliated coordinates for χ around $(x_0, t_0) \in M \times [0, +\infty)$, such that g_f takes a diagonal expression with eigenvalues $\lambda_1, \dots, \lambda_n$, one has:

$$\begin{aligned} \det(\chi) = 1, \quad \det(g_f^T) = \lambda_1 \cdots \lambda_n, \quad \sigma_f = g_f^{\bar{j}k} \chi_{j\bar{k}} = \sum_{j=1}^n \frac{1}{\lambda_j}, \quad \gamma_f = \chi^{\bar{j}k} (g_f)_{j\bar{k}} = \sum_{j=1}^n \lambda_j, \\ g_f^{\bar{q}p} g_f^{\bar{p}s} (g_f)_{s\bar{q}} = \sum_{j=1}^n \frac{1}{\lambda_j} = \sigma_f, \quad g_f^{\bar{q}p} g_f^{\bar{p}s} \chi_{s\bar{q}} = \sum_{j=1}^n \frac{1}{\lambda_j^2}. \end{aligned}$$

3. SECOND ORDER ESTIMATES

In order to develop the second order estimates we begin with the following lemma:

Lemma 3.1. *Let (M, g, ξ, Φ, η) be a $(2n+1)$ -dimensional Sasakian manifold and let f be a solution to (2) in $[0, +\infty)$. Then at any point $(x_0, t_0) \in M \times [0, +\infty)$:*

$$(\tilde{\Delta}_f - \partial_t)(\log \gamma_f) \geq \frac{1}{\gamma_f} g_f^{\bar{q}p} g_f^{\bar{p}s} R(\chi)^{\bar{b}a}_{s\bar{q}} (g_f)_{a\bar{b}} - \frac{1}{\gamma_f} g_f^{\bar{k}j} \text{Ric}(\chi)_{j\bar{k}}.$$

Proof. Compute:

$$\partial_t \log(\gamma_f) = \frac{1}{\gamma_f} \chi^{\bar{j}k} \partial_t [(g_f)_{k\bar{j}}] = \frac{1}{\gamma_f} \chi^{\bar{j}k} \dot{f}_{,k\bar{j}},$$

where:

$$\begin{aligned} \dot{f}_{,a\bar{b}} = & -2g_f^{\bar{k}s} (g_f)_{\bar{r}s,\bar{b}} g_f^{\bar{r}p} (g_f)_{p\bar{q},a} g_f^{\bar{q}j} \chi_{j\bar{k}} + g_f^{\bar{q}p} g_f^{\bar{r}s} \chi_{p\bar{r}} (g_f)_{a\bar{b},s\bar{q}} \\ & + g_f^{\bar{k}p} (g_f)_{p\bar{q},a} g_f^{\bar{q}j} \chi_{j\bar{k},\bar{b}} + g_f^{\bar{k}s} (g_f)_{\bar{r}s,\bar{b}} g_f^{\bar{r}j} \chi_{j\bar{k},a} - g_f^{\bar{k}j} \chi_{j\bar{k},a\bar{b}}. \end{aligned}$$

Thus:

$$\begin{aligned} \partial_t \log(\gamma_f) = & \frac{1}{\gamma_f} \chi^{\bar{b}a} \left(-2g_f^{\bar{k}s} (g_f)_{\bar{r}s,\bar{b}} g_f^{\bar{r}p} (g_f)_{p\bar{q},a} g_f^{\bar{q}j} \chi_{j\bar{k}} + g_f^{\bar{q}p} g_f^{\bar{r}s} \chi_{p\bar{r}} (g_f)_{a\bar{b},s\bar{q}} \right. \\ & \left. + g_f^{\bar{k}p} (g_f)_{p\bar{q},a} g_f^{\bar{q}j} \chi_{j\bar{k},\bar{b}} + g_f^{\bar{k}s} (g_f)_{\bar{r}s,\bar{b}} g_f^{\bar{r}j} \chi_{j\bar{k},a} - g_f^{\bar{k}j} \chi_{j\bar{k},a\bar{b}} \right). \end{aligned}$$

Taking special coordinates around (x_0, t_0) we get:

$$\partial_t \log(\gamma_f) = \frac{1}{\gamma_f} \chi^{\bar{b}a} \left(-2g_f^{\bar{k}s} (g_f)_{\bar{r}s,\bar{b}} g_f^{\bar{r}p} (g_f)_{p\bar{q},a} g_f^{\bar{q}j} \chi_{j\bar{k}} + g_f^{\bar{q}p} g_f^{\bar{r}s} \chi_{p\bar{r}} (g_f)_{a\bar{b},s\bar{q}} \right) + \frac{1}{\gamma_f} g_f^{\bar{k}j} \text{Ric}(\chi)_{j\bar{k}}.$$

Further:

$$\tilde{\Delta}_f[\log \gamma_f] = g_f^{\bar{q}p} g_f^{\bar{r}s} \chi_{p\bar{r}}(\log \gamma_f)_{,s\bar{q}} = g_f^{\bar{q}p} g_f^{\bar{r}s} \chi_{p\bar{r}}(-\gamma_f^{-2}(\gamma_f)_{,\bar{q}}(\gamma_f)_{,s} + \gamma_f^{-1}(\gamma_f)_{,s\bar{q}}),$$

where at (x_0, t_0) :

$$\begin{aligned} (\gamma_f)_{,q} &= -\chi^{\bar{b}r} \chi_{\bar{r}s,q} \chi^{\bar{s}a} (g_f)_{a\bar{b}} + \chi^{\bar{b}a} (g_f)_{a\bar{b},q} = \chi^{\bar{b}a} (g_f)_{a\bar{b},q}, \\ (\gamma_f)_{,s\bar{q}} &= -\chi^{\bar{b}r} \chi_{\bar{r}j,s\bar{q}} \chi^{\bar{j}a} (g_f)_{a\bar{b}} + \chi^{\bar{b}a} (g_f)_{a\bar{b},s\bar{q}}. \end{aligned}$$

Thus:

$$\tilde{\Delta}_f[\log \gamma_f] = -\frac{1}{\gamma_f^2} g_f^{\bar{q}p} g_f^{\bar{p}s} (g_f)_{a\bar{a},\bar{q}} (g_f)_{b\bar{b},s} + \frac{1}{\gamma_f} g_f^{\bar{q}p} g_f^{\bar{p}s} R(\chi)^{\bar{b}a}_{s\bar{q}} (g_f)_{a\bar{b}} + \frac{1}{\gamma_f} g_f^{\bar{q}p} g_f^{\bar{p}s} \chi^{\bar{b}a} (g_f)_{a\bar{b},s\bar{q}},$$

which implies:

$$(4) \quad (\tilde{\Delta}_f - \partial_t)(\log \gamma_f) \geq \frac{1}{\gamma_f} g_f^{\bar{q}p} g_f^{\bar{p}s} R(\chi)^{\bar{b}a}_{s\bar{q}} (g_f)_{a\bar{b}} - \frac{1}{\gamma_f} g_f^{\bar{k}j} \text{Ric}(\chi)_{j\bar{k}},$$

where we used that by [24, Lemma 3.2], one has:

$$\gamma_f \chi^{\bar{b}a} g_f^{\bar{k}s} g_f^{\bar{r}p} g_f^{\bar{q}j} \chi_{j\bar{k}} (g_f)_{\bar{r}s,\bar{b}} (g_f)_{p\bar{q},a} \geq \chi^{\bar{b}a} \chi^{\bar{k}j} g_f^{\bar{q}p} g_f^{\bar{r}s} \chi_{p\bar{r}} (g_f)_{a\bar{b},\bar{q}} (g_f)_{j\bar{k},s}.$$

□

Recall now that a uniform lower bound on the second derivatives of a solution f to the Sasaki J -flow is obtained in [23, Lemma 6.1]. In order to get a uniform upper bound, we start proving the following proposition, which follows essentially [25, Th. 2.1] (see also [24, Th. 3.1] for the case $n = 2$).

Proposition 3.2. *Let (M, g, ξ, Φ, η) be a $2n + 1$ -dimensional Sasakian manifold and let f be a solution to (2) in $[0, +\infty)$. Assume that $\frac{\epsilon}{2}d\eta - (n-1)\chi > 0$. Then, for any $t \geq 0$ there exist constants A and C , depending only on the initial data, such that $\gamma_f \leq C e^{A(f - \inf_{M \times [t,0]} f)}$ in $[0, t]$.*

Proof. Normalize χ in order to get $c = 1$ (i.e. $\frac{1}{2}d\eta - (n-1)\chi > 0$). Fix $t > 0$ and let (x_0, t_0) be a maximum in $M \times [0, t]$ for $\log \gamma_f - Af$, where A is a constant to be fix later. By Lemma 3.1 above at (x_0, t_0) we get:

$$(\tilde{\Delta}_f - \partial_t)(\log \gamma_f) \geq \frac{1}{\gamma_f} g_f^{\bar{q}p} g_f^{\bar{p}s} R(\chi)^{\bar{b}a}_{s\bar{q}} (g_f)_{a\bar{b}} - \frac{1}{\gamma_f} g_f^{\bar{k}j} \text{Ric}(\chi)_{j\bar{k}}.$$

Further, we have:

$$(5) \quad (\tilde{\Delta}_f - \partial_t)f = g_f^{\bar{q}p} g_f^{\bar{r}s} \chi_{p\bar{r}} f_{s\bar{q}} - \dot{f} = g_f^{\bar{q}p} g_f^{\bar{p}s} (g_f)_{s\bar{q}} - g_f^{\bar{q}p} g_f^{\bar{p}s} (g_0)_{s\bar{q}} - \dot{f}.$$

Thus, by Remark 2.2 and since with our normalization a solution f to (2) satisfies $\dot{f} = 1 - \sigma_f$, it follows:

$$(\tilde{\Delta}_f - \partial_t)(\log \gamma_f - Af) \geq \frac{1}{\gamma_f} g_f^{\bar{q}p} g_f^{\bar{p}s} R(\chi)^{\bar{b}a}_{s\bar{q}} (g_f)_{a\bar{b}} - \frac{1}{\gamma_f} g_f^{\bar{k}j} \text{Ric}(\chi)_{j\bar{k}} - 2A\sigma_f + Ag_f^{\bar{q}p} g_f^{\bar{p}s} (g_0)_{s\bar{q}} + A.$$

Let C_0 be a positive constant such that:

$$R(\chi)^{\bar{b}a}_{j\bar{k}} \geq -C_0 \chi^{\bar{b}a} (g_0)_{s\bar{q}},$$

From $\frac{1}{2}d\eta - \chi > 0$ it follows that we can choose $\epsilon > 0$ small enough to have:

$$(6) \quad \frac{1}{2}d\eta \geq (n-1 + (n+1)\epsilon)\chi,$$

and we can set A big enough such that:

$$\epsilon A g_f^{\bar{q}p} g_f^{\bar{p}s}(g_0)_{s\bar{q}} \geq -C_0 g_f^{\bar{q}p} g_f^{\bar{p}s}(g_0)_{s\bar{q}} - \frac{1}{\gamma_f} g_f^{\bar{k}j} \text{Ric}(\chi)_{j\bar{k}}.$$

Thus:

$$(\tilde{\Delta}_f - \partial_t)(\log \gamma_f - Af) \geq A \left((1-\epsilon) g_f^{\bar{q}p} g_f^{\bar{p}s}(g_0)_{s\bar{q}} - 2\sigma_f + 1 \right).$$

Since at (x_0, t_0) it follows easily by (6) that one has:

$$(1-\epsilon)(g_0)_{s\bar{q}} \geq (n-1+\epsilon)\chi_{s\bar{q}},$$

by Remark 2.2 we finally get:

$$(\tilde{\Delta} - \partial_t)(\log \gamma_f - Af) \geq A \left((n-1+\epsilon) \sum_{j=1}^n \frac{1}{\lambda_j^2} - 2 \sum_{j=1}^n \frac{1}{\lambda_j} + 1 \right).$$

At this point, observe that (x_0, t_0) has been chosen to be a global maximum in $M \times [0, t]$ and thus (see Remark 2.1):

$$0 \geq (\tilde{\Delta} - \partial_t)(\log \gamma_f - Af),$$

which implies:

$$(n-1+\epsilon) \sum_{j=1}^n \frac{1}{\lambda_j^2} - 2 \sum_{j=1}^n \frac{1}{\lambda_j} + 1 \leq 0.$$

This last inequality implies an upper bound for all λ_j , as it follows considering that we can rewrite it as:

$$\sum_{j=1}^n \left(\frac{1}{\sqrt{n-1+\epsilon}} - \frac{\sqrt{n-1+\epsilon}}{\lambda_j} \right)^2 - \frac{n}{n-1+\epsilon} + 1 \leq 0,$$

and thus for any $j = 1, \dots, n$:

$$\frac{1}{\sqrt{n-1+\epsilon}} - \frac{\sqrt{n-1+\epsilon}}{\lambda_j} \leq \frac{\sqrt{1-\epsilon}}{\sqrt{n-1+\epsilon}},$$

i.e.:

$$\lambda_j \leq \frac{n-1+\epsilon}{1-\sqrt{1-\epsilon}}.$$

It follows that at (x_0, t_0) , γ_f is bounded above. Since (x_0, t_0) is the global maximum in $[0, t]$ for $\log \gamma_t - Af$, we get that

$$\log \gamma_t - Af \leq \log C - A \inf_{M \times [t, 0]} f,$$

i.e.:

$$\gamma_t \leq C e^{A(f - \inf_{M \times [0, t]} f)},$$

as wished. \square

4. C^0 ESTIMATES AND THE PROOF OF THEOREM 1.1

In order to get a uniform upper bound for a solution to (2) we modify the arguments in [24, 25]. For this purpose, let g_χ be the Riemannian metric which has χ as transverse Kähler metric, i.e.:

$$g_\chi(\cdot, \cdot) = \chi(\cdot, \Phi \cdot) + \eta(\cdot) \eta(\cdot).$$

Observe that since (M, g_χ) is a compact Riemannian manifold, there exists a Green function $G(x, y)$ which satisfies for any $u \in C^\infty(M)$:

$$u(x) = \int_M G(x, y) \Delta u(y) d\mu(y) + \frac{1}{\int_M d\mu} \int_M u d\mu,$$

where $d\mu$ and Δ are respectively the volume form and the Riemannian Laplacian associated to g_χ . By [17, Prop. 2.8] $\Delta_\chi \psi = -\Delta \psi$ for any $\psi \in C_B^\infty(M, \mathbb{R})$, where Δ_χ is the basic Laplacian associated to χ , i.e. it is locally expressed by:

$$\Delta_\chi \psi = \chi^{\bar{j}r} \psi_{,r\bar{j}}, \quad \text{for } \psi \in C_B^\infty(M, \mathbb{R}),$$

(in our notation the basic Laplacian has the opposite sign of [17] one). Thus, for any $\psi \in C_B^\infty(M, \mathbb{R})$ we have:

$$(7) \quad \psi(x) = - \int_M G(x, y) \Delta_\chi \psi(y) d\mu + \frac{1}{\int_M d\mu} \int_M \psi d\mu.$$

Remark 4.1. Notice that $\Delta_\chi f$ is uniformly bounded from below, as it follows easily from the definition of $\Delta_\chi f$ and by observing that:

$$\chi^{\bar{j}k} (g_f^T)_{j\bar{k}} = \chi^{\bar{j}k} ((g^T)_{j\bar{k}} + f_{,j\bar{k}}) > 0.$$

Proposition 4.2. *Let (M, g, ξ, Φ, η) be a $(2n+1)$ -dimensional Sasakian manifold and let f be a solution to (2). Then there exist two positive constants C_0 and C_1 , depending only on the initial data, such that:*

$$0 \leq \sup_M f \leq C_0 - C_1 \inf_M f.$$

Proof. Observe first that from $f \in \mathcal{H}_0$ by (1) we get:

$$(8) \quad \sum_{p=0}^n \frac{n!}{(p+1)!(n-p)!} \int_M f \eta \wedge (d\eta)^{n-p} \wedge (i\partial_B \bar{\partial}_B f)^p = 0.$$

Thus f vanishes somewhere and we have $\sup_M f \geq 0$.

In order to prove the second inequality, let B_0, B_1 be constants such that:

$$d\mu \leq B_0 \eta \wedge (d\eta)^n, \quad d\eta \leq B_1 \chi.$$

From (8) we get:

$$\int_M f \eta \wedge (d\eta)^n = -n \int_M \eta \wedge (d\eta_f)^{n-1} \wedge (i\partial_B \bar{\partial}_B f) = -n \int_M \eta \wedge (d\eta_f)^n + n \int_M \eta \wedge d\eta \wedge (d\eta_f)^{n-1},$$

and thus:

$$\begin{aligned}
\int_M f d\mu &\leq B_0 \int_M f \eta \wedge (d\eta)^n \\
&= -nB_0 \int_M f \eta \wedge (d\eta_f)^n + nB_0 \int_f \eta \wedge d\eta \wedge (d\eta_f)^{n-1} \\
&\leq -nB_0 \int_M f \eta \wedge (d\eta_f)^n + nB_0 B_1 \int_f \eta \wedge \chi \wedge (d\eta_f)^{n-1} \\
&= -nB_0 \int_M (f - \inf_M f) \eta \wedge (d\eta_f)^n - nB_0 \inf_M f \int_M \eta \wedge (d\eta)^n + nB_0 B_1 \int_f \eta \wedge \chi \wedge (d\eta)^{n-1} \\
&\leq nB_0 B_1 \int_f \eta \wedge \chi \wedge (d\eta_f)^{n-1} - nB_0 \int_M \eta \wedge (d\eta)^n \inf_M f.
\end{aligned}$$

Thus, by (7):

$$f(x) \leq - \int_M G(x, y) \Delta_\chi f(y) \eta \wedge (d\eta)^n + nB_0 B_1 \int_f \eta \wedge \chi \wedge (d\eta)^{n-1} - nB_0 \int_M \eta \wedge (d\eta)^n \inf_M f,$$

and conclusion follows by the existence of a lower bound for the Green function of g and from Remark 4.1. \square

It remains to prove that $\inf_M f$ is uniformly bounded from above. Following [25], assume that such bound does not exist. Then there exists a sequence of time t_i such that $t_i \rightarrow \infty$ implies $\inf_{t_i} \inf_M f \rightarrow \infty$. Fix i and set $\psi_i(x) = f(x, t_i) - \sup_M f(x, t_i)$. Since by Prop. 4.2 above $\sup_M f \geq 0$, we have $\sup_M \psi_i = 0$. This last fact, together with Prop. 3.2 in the previous section, will lead us with Prop. 4.6 to the contradiction $\|e^{-B\psi_i}\|_{C^0} < 1$, where $B = A/(4 - \epsilon)$ for a small $\epsilon > 0$ which is set in the proof of Prop. 4.6.

We begin with the following lemma.

Lemma 4.3. *Let (M, g, ξ, Φ, η) be a compact Sasaki $2n + 1$ -dimensional manifold and let χ a transverse Kähler form on M and $a > 0$. If $\psi \in \mathcal{H}$ satisfies $\gamma_\psi \leq Ce^{A(\psi - \inf_{M \times [t, 0]} \psi)}$ for some constants A and C , then:*

$$\int_M |\nabla e^{-a\psi}|^2 \eta \wedge \chi^n \leq \frac{aC}{2} e^{-A \inf_{M \times [t, 0]} \psi} \int_M e^{(A-2a)\psi} \eta \wedge \chi^n.$$

Proof. Observe first that:

$$(9) \quad \int_M |\nabla e^{-a\psi}|^2 \eta \wedge \chi^n = (\partial_B e^{-a\psi}, \partial_B e^{-a\psi})_\chi = (\partial_B^* \partial_B e^{-a\psi}, e^{-a\psi})_\chi = -(\Delta_\chi e^{-a\psi}, e^{-a\psi})_\chi,$$

From:

$$\Delta_\chi e^{-a\psi} = \chi^{\bar{k}j} \left(e^{-a\psi} \right)_{,j\bar{k}} = \chi^{\bar{k}j} \left(-ae^{-a\psi} \psi_{,j\bar{k}} + a^2 e^{-a\psi} \psi_{,j} \psi_{,\bar{k}} \right),$$

it follows that:

$$(\Delta_\chi e^{-a\psi}, e^{-a\psi})_\chi = -a(\Delta_\chi \psi, e^{-2a\psi})_\chi + a^2 (\partial_B \psi, e^{-2a\psi} \partial_B \psi)_\chi,$$

and since

$$-2a e^{-2a\psi} \partial_B \psi = \partial_B e^{-2a\psi},$$

we get:

$$(\Delta_\chi e^{-\alpha\psi}, e^{-\alpha\psi})_\chi = -a(\Delta_\chi \psi, e^{-2\alpha\psi})_\chi - \frac{a}{2}(\partial_B \psi, \partial_B e^{-2\alpha\psi})_\chi = -\frac{a}{2}(\Delta_\chi \psi, e^{-2\alpha\psi})_\chi.$$

Thus, plugging this last equality into (9) we get:

$$\int_M |\nabla e^{-\alpha\psi}|^2 \eta \wedge \chi^2 = \frac{a}{2}(\Delta_\chi \psi, e^{-2\alpha\psi})_\chi.$$

Since $\chi^{j\bar{k}}(g_0^T)_{j\bar{k}} \geq 0$ and we assumed $\gamma_\psi \leq C e^{A(\psi - \inf_{M \times [0, t]} \psi)}$, conclusion follows by observing that:

$$\Delta_\chi \psi = \gamma_\psi - \chi^{j\bar{k}}(g_0^T)_{j\bar{k}}.$$

□

The next lemma is a Sasakian version of [21, Prop. 2.1].

Lemma 4.4. *Let (M, g, ξ, Φ, η) be a compact Sasaki $2n + 1$ -dimensional manifold and let χ be a second transverse Kähler form on M . Assume that $\psi \in \mathcal{H}$ satisfies $\sup_M \psi = 0$. Then for a small enough $\alpha > 0$, there exists a constant C' depending only on the initial data, such that:*

$$\int_M e^{-\alpha\psi} \eta \wedge \chi^n \leq C'.$$

Proof. By (7) we have:

$$0 = \sup_M \psi(x) \leq \frac{1}{\int_M d\mu} \int_M \psi d\mu + \sup_M \int_M G(x, y)(-\Delta_\chi \psi(y)) d\mu,$$

which implies:

$$\frac{1}{\int_M d\mu} \int_M \psi d\mu \geq \sup_M \int_M G(x, y)(-\Delta_\chi \psi(y)) d\mu \geq -B_1,$$

i.e., for some constant B_2 :

$$\int_M \psi \eta \wedge \chi^n \geq -B_2.$$

At this point, recall that locally M can be described by special coordinates $(z_1, \dots, z_n, z) \in \mathbb{C}^n \times \mathbb{R}$ and being ψ basic, it is constant in z . Let $\{B_{r_i}\}_i$ be geodesic balls that cover M . For each i , set special coordinates on B_{r_i} . Since the support of the smooth function $\psi' = \psi - \psi(0)$ is contained in $\tilde{B}_{r_i} = \{(z_1, \dots, z_n, z) \in B_{r_i} \mid z = 0\}$, we can apply the same method as in [21, Prop. 2.1] to get the existence of a constant B_3 such that:

$$\int_M e^{-\alpha\psi'} \eta \wedge \chi^n \leq B_3.$$

Then we have:

$$\int_M e^{-\alpha\psi} \eta \wedge \chi^n \leq B_3 e^{-\alpha\psi(0)},$$

and the assertion follows setting $C' = B_3 e^{-\alpha\psi(0)}$.

□

We recall here a Sobolev inequality for a compact Riemannian manifold (M, g) needed in the proof of Prop. 4.6 below. For any smooth function h on M and $a > 0$ denote:

$$||h||_a = \left(\int_M |h|^a d\mu_g \right)^{\frac{1}{a}},$$

where $d\mu_g$ is the volume form associated to g . Then we have the following (see e.g. [12, Th. 2.6]):

Theorem 4.5 (Sobolev inequality). *Let (M, g) be a smooth compact Riemannian manifold of dimension m . Then for any real numbers $1 \leq q < p$ with $1/p = 1/q - 1/m$, there exists a constant C_1 such that:*

$$||h||_p \leq C_1 (||\nabla h||_q + ||h||_q).$$

In particular, setting $q = 2$, $m = 2n + 1$ and raising to the second power, there exists a constant C_1 such that:

$$(10) \quad ||h||_b^2 \leq C_1 (||\nabla h||_2^2 + ||h||_2^2),$$

for $b = 2(2n + 1)/(2n - 1)$.

Proposition 4.6. *Let (M, g, ξ, Φ, η) be a compact Sasaki $2n + 1$ -dimensional manifold and let χ be a second transverse Kähler form on M . If $\psi \in \mathcal{H}$ satisfies:*

- (i) $\sup_M \psi = 0$,
- (ii) $\gamma_\psi \leq C e^{A(\psi - \inf_{M \times [t, 0]} \psi)}$,

then for some small $\epsilon > 0$, $||e^{-\frac{A}{4-\epsilon}\psi}||_{C^0} \leq C'$, where C' is a constant depending on A , C and the initial data.

Proof. Since (M, g) is a Riemannian manifold, then by (10), for $b = 2(2n + 1)/(2n - 1)$ and some constant C_1 , the following Sobolev inequality holds for any $u \in C^\infty(M)$:

$$||u||_b^2 \leq C_1 (||\nabla u||_2^2 + ||u||_2^2),$$

i.e., since the Sobolev inequality is independent from the volume form chosen:

$$\left(\int_M |u|^b \eta \wedge \chi^n \right)^{\frac{2}{b}} \leq C_1 \left(\int_M |\nabla u|^2 \eta \wedge \chi^n + \int_M |u|^2 \eta \wedge \chi^n \right).$$

For $0 < q < p$, set $u = e^{-\frac{A}{3p+q}\psi}$. Since $\int_M |u|^2 \eta \wedge \chi^n \geq 0$, by Lemma 4.3 applied to u^{2p} , there exists a constant C_1 such that:

$$\begin{aligned} \left(\int_M |e^{-\frac{2pA}{3p+q}\psi}|^b \eta \wedge \chi^n \right)^{\frac{2}{b}} &\leq C_1 \left(\int_M |\nabla e^{-\frac{2pA}{3p+q}\psi}|^2 \eta \wedge \chi^n + \int_M |e^{-\frac{2pA}{3p+q}\psi}|^2 \eta \wedge \chi^n \right) \\ &\leq \frac{C_1 C p A}{3p + q} e^{-A \inf_{M \times [t, 0]} \psi} \int_M e^{-\frac{A}{3p+q}(p-q)\psi} \eta \wedge \chi^n \end{aligned}$$

Then, raising to the power of $1/4p$, we get

$$(11) \quad ||u||_{2bp} \leq \left(\frac{C_1 C p A}{3p + q} \right)^{\frac{1}{4p}} e^{-\frac{A}{4p} \inf_{M \times [t, 0]} \psi} ||u||_{\frac{p-q}{4p}}$$

Now set:

$$s_0 = p, \quad s_k = p(2b)^k + q \left((2b)^{k-1} + \dots + 2b + 1 \right) \quad \text{for } k = 1, 2, \dots$$

With this notation (11) reads:

$$(12) \quad \|u\|_{2bs_0} \leq (C_1 C A)^{\frac{1}{4s_0}} \left(\frac{s_0}{3s_0 + q} \right)^{\frac{1}{4s_0}} e^{-\frac{A}{4s_0} \inf_{M \times [t, 0]} \psi} \|u\|_{\frac{p-q}{4p}}^{\frac{p-q}{4p}}.$$

Replacing p with $2bp + q$, s_k change in s_{k+1} for all $k = 0, 1, 2, \dots$ and we get:

$$\|u\|_{2bs_1} \leq (C_1 C A)^{\frac{1}{4s_1}} \left(\frac{s_1}{3s_1 + q} \right)^{\frac{1}{4s_1}} e^{-\frac{A}{4s_1} \inf_{M \times [t, 0]} \psi} (\|u\|_{2bs_0})^{\frac{bs_0}{2s_1}},$$

and thus by (12):

$$\|u\|_{2bs_1} \leq (C_1 C A)^{\frac{1}{4s_1}(1+\frac{b}{2})} \left(\frac{s_1}{3s_1 + q} \right)^{\frac{1}{4s_1}} \left(\frac{s_0}{3s_0 + q} \right)^{\frac{b}{2} \frac{1}{4s_1}} e^{-\frac{A}{4s_1}(1+\frac{b}{2}) \inf_{M \times [t, 0]} \psi} \left(\|u\|_{\frac{p-q}{4p}}^{\frac{p-q}{4p}} \right)^{\frac{bs_0}{2s_1}},$$

If we iterate this procedure k times, we get:

$$\|u\|_{2bs_k} \leq (C_1 C A)^{a_k} \left(\prod_{j=0}^k \left(\frac{s_j}{3s_j + q} \right)^{\left(\frac{b}{2}\right)^{k-j}} \right)^{\frac{1}{4s_k}} e^{-Aa_k \inf_{M \times [0, t]} \psi} \left(\|u\|_{\frac{p-q}{4p}}^{\frac{p-q}{4p}} \right)^{\frac{b^k s_0}{2^k s_k}},$$

where we set $a_k = \frac{1}{4s_k} \sum_{j=0}^k \left(\frac{b}{2}\right)^k$. Observe now that, since $b = 2(2n+1)/(2n-1)$ implies $b/2 > 1$ and $s_k \rightarrow +\infty$ as k approaches infinity, we have:

$$\lim_{k \rightarrow +\infty} a_k = \lim_{k \rightarrow +\infty} \left(\frac{1}{4s_k} \sum_{j=0}^k \left(\frac{b}{2}\right)^k \right) = 0, \quad \lim_{k \rightarrow +\infty} \left(\frac{b^k s_0}{2^k s_k} \right) = 0;$$

further, since for any $j = 0, \dots, k$,

$$\lim_{k \rightarrow +\infty} \left(\frac{1}{s_k} \left(\frac{b}{2}\right)^{k-j} \right) = 0, \quad \frac{p}{3p+q} \leq \frac{s_j}{3s_j+q} \leq \frac{1}{3}$$

we have:

$$\lim_{k \rightarrow +\infty} \left(\prod_{j=0}^k \left(\frac{s_j}{3s_j + q} \right)^{\left(\frac{b}{2}\right)^{k-j}} \right)^{\frac{1}{4s_k}} = 1.$$

Setting $p = 1$ and $q = 1 - \epsilon$, by Lemma 4.4 there exists a small enough $\epsilon > 0$ such that:

$$\|u\|_{p-q} = \|u\|_{\epsilon} = \int_M e^{-\frac{A\epsilon}{4-\epsilon} \psi} \eta \wedge \chi^n,$$

is bounded and the bound on $\|e^{-\frac{A}{4-\epsilon} \psi}\|_{C^0}$ follows readily. \square

Corollary 4.7. *Let $\frac{\epsilon}{2} d\eta - (n-1)\chi > 0$. Then the second order derivatives of a solution f to the Sasaki J -flow are uniformly bounded from above.*

Proof. By Prop. 4.6 and the discussion above, $\inf_M f$ is uniformly bounded from above. The bound on the second order derivatives of f follows then by Prop. 3.2 and Prop. 4.2. \square

We are now in the position of proving Theorem 1.1.

Proof of Theorem 1.1. By [23, Th. 1.1], there exists a solution $f : M \times [0, +\infty) \rightarrow \mathbb{R}$ to the Sasaki J -flow. By [23, Lemma 6.1] and by Corollary 4.7 above, the second derivatives $\partial_j \partial_{\bar{k}} f$ are uniformly bounded. Since a solution to the Sasaki J -flow can be regarded as a solution to the Kähler J -flow on small open balls of \mathbb{C}^n , by [23, Th. 7.1] we get uniform C^∞ bounds on f . Then, by Ascoli-Arzelà Theorem, given a sequence $t_j \in [0, \infty)$, $t_j \rightarrow \infty$, there exists a subsequence f_{t_j} converging in C^∞ -norm to a function f_∞ as $t_j \rightarrow \infty$.

At this point, observe that \dot{f} satisfies the heat equation $\partial_t \dot{f} = -\tilde{\Delta} \dot{f}$ and we have uniform bounds for $(g_f^T)_{j\bar{k}}$, $(\partial_t g_f^T)_{j\bar{k}}$, and all the covariant derivatives of $(g_f^T)_{j\bar{k}}$ and for $(\partial_t g_f^T)_{j\bar{k}}$. Then we get uniform bounds also for the family g_t , $t \in [0, +\infty)$, of Riemannian metric on M defined by:

$$g_t(\cdot, \cdot) = g_f^T(\cdot, \cdot) + \eta(\cdot)\eta(\cdot),$$

and for all its covariant derivatives. Thus we can apply the argument in [4] (Th. 2.1 and discussion below) to get:

$$\sup_M f - \inf_M f \leq C_0 e^{-C_1 t}.$$

for some constant C_0 and C_1 independent of t . The convergence of f in the C^∞ topology follows by the same argument as in [24, Sec. 5]. \square

5. MABUCHI K -ENERGY AND THE J -FLOW

Let $\bar{\mathcal{H}}$ be the completion of \mathcal{H} with respect to the C_w^2 -norm. In [11], P. Guan, X. Zhang proved that any two points in \mathcal{H} can be connected by a $C^{1,1}$ -geodesic. By definition, a $C^{1,1}$ -geodesic is a curve in $\bar{\mathcal{H}}$ obtained as weak limit of solutions to:

$$\left(\ddot{f} - \frac{1}{4} |d_B \dot{f}|_f^2 \right) \eta \wedge (d\eta_f)^n = \epsilon \eta \wedge (d\eta)^n.$$

This result allows us to prove the following (cfr. [5, Prop. 3]).

Proposition 5.1. *If there exists a critical metric then the functional $J_\chi : \mathcal{H}_0 \rightarrow \mathbb{R}$ is uniformly bounded from below.*

Proof. Let f_∞ be a critical point in \mathcal{H}_0 , $f_1 \in \mathcal{H}_0$ and let $f : [0, 1] \rightarrow \bar{\mathcal{H}}$ be a $C^{1,1}$ -geodesic such that $f(0) = f_\infty$, $f(1) = f_1$. Then by the estimates in the proof of Prop. 3.2 in [23], J_χ satisfies:

$$\partial_t^2 J_\chi(f) \geq 0.$$

Further, being f_∞ a critical point, $\partial_t J_\chi(f)|_{t=0} = 0$. Thus, we have $J_\chi(f_1) \geq J_\chi(f_\infty)$ for any $f_1 \in \mathcal{H}_0$. \square

Let α be a transverse Kähler form on M and denote by $[\alpha]_B$ the basic $(1, 1)$ class associated to α . Define:

$$\mathcal{K} = \{\text{transverse Kähler form in the basic } (1, 1) \text{ class } [d\eta]_B\},$$

and observe that $\mathcal{H}_0 \simeq \mathcal{K}$. Further, denote by s_f^T the transverse scalar curvature associated to $d\eta_f$, namely in local coordinates $s_f^T = (g_f^T)^{\bar{k}j} \text{Ric}(d\eta_f)_{j\bar{k}}$, where $\text{Ric}(d\eta_f)$ is the Ricci tensor

associated to $d\eta_f$. Let us also denote by ρ^T the transverse Ricci form associated to $\text{Ric}(d\eta)$ and by \bar{s}^T the average scalar curvature defined by:

$$\bar{s}^T = \frac{2n \int_M \rho^T \wedge \eta \wedge (d\eta)^{n-1}}{\int_M \eta \wedge (d\eta)^n}.$$

The Mabuchi K -energy in the Sasakian context has been introduced by A. Futaki, H. Ono and G. Wang in [9]. According to the notation in [14], it is defined as follows. Let $f_0, f_1 \in \mathcal{H}$ and $f: [0, 1] \rightarrow \mathcal{H}$ be a smooth path satisfying $f(0) = f_0$, $f(1) = f_1$. Then the functional $\mathcal{M}: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ defined by:

$$\mathcal{M}(f_0, f_1) := \int_0^1 \int_M \dot{f} (s_f^T - \bar{s}^T) \wedge \eta \wedge (d\eta_f)^n dt,$$

is well defined and factors through $\mathcal{H}_0 \times H_0$ (see e.g. [14, Lemma 3.2]). Define the K -energy map of the transverse Kähler class $[d\eta]_B$ by $\mathcal{M}: \mathcal{K} \rightarrow \mathbb{R}$, $\mathcal{M}(d\eta_f) = \mathcal{M}(d\eta, d\eta_f)$. Further, the map $\mathcal{M}: \mathcal{H} \rightarrow \mathbb{R}$, $\mathcal{M}(f) = \mathcal{M}(0, f)$, is called the K -energy map of \mathcal{H} .

We can prove now our second result, which should be compared with [5, 24, 25].

Theorem 5.2. *Let (M, ξ, Φ, η, g) be a Sasakian manifold and assume that $-\rho^T$ is a positive transverse Kähler form. If:*

$$(13) \quad \frac{\bar{s}^T}{2} [d\eta]_B + (n-1) [\rho^T]_B > 0,$$

then the Mabuchi K -energy is bounded below on $[d\eta]_B$.

Proof. Define a J_χ functional with $\chi = -\rho^T$. By Prop. 5.1 and Theorem 1.1, condition (13) implies that this functional is bounded from below. Conclusion follows by observing that the Sasakian version of the Mabuchi K -energy map can be written as (see [14, Prop. 3.2]):

$$\mathcal{M}(f) = \frac{\bar{s}^T}{n+1} I(f) + 2J_{-\rho^T}(f) + 2 \int_M \ln \left(\frac{\eta \wedge (d\eta_f)^n}{\eta \wedge (d\eta)^n} \right) \eta \wedge (d\eta_f)^n,$$

where from $f \in \mathcal{H}_0$ follows $I(f) = 0$, and since $x \ln x > -e^{-1}$ for any $x > 0$, the term $\int_M \ln \left(\frac{\eta \wedge (d\eta_f)^n}{\eta \wedge (d\eta)^n} \right) \eta \wedge (d\eta_f)^n$ is bounded below by $-e^{-1} \int_M \eta \wedge (d\eta)^n$. \square

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